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Averages: There is Still Something to Learn

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Abstract
The common way to deal with outliers in empirical Economics and Finance is to delete them, either by trimming or winsorizing, or by computing statistics robust to outliers. However, due to their importance, there are situations where the exclusion of these observations is not reasonable and may even be counterproductive. For example, should we exclude the very high stock prices of Amazon and Google from an empirical analysis? Even if the purpose is to compute an average of tech stock prices, does it make economic and financial sense? Maybe not. A solution that would keep the two companies in the data set and yet not penalize the higher observations as much as the median, harmonic and geometric averages, might—were such a solution to be available—constitute an attractive alternative. In this paper we propose and analyze a modified measure, the adjusted median, where the influence of the outlying observations, while not as high as in the arithmetic average would, however, give more weight to the outlying observations than the median, harmonic and geometric averages. Monte Carlo simulations and bootstrapping real financial data confirm how useful the adjusted median could be.

Keywords Adjusted median · Central tendency · Monte Carlo simulation · Bootstrapping

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1 Introduction

The traditional ways to deal with outlying observations in empirical Economics and Finance is to exclude them (by trimming or winsorizing), or by computing statistics robust to outliers: the median and inter-quartile range, for example. Due to their importance, however, there are situations where the exclusion of the observations is not reasonable and may even be counterproductive. Suppose that we compute the average of stock prices of companies listed in the “Information Technology” sector of Standard and Poor’s 500 (S&P 500), including Amazon, Google, Microsoft and Apple, among many others. As the stock prices of Amazon and Google are much higher in comparison to the others, would it make sense to exclude these two companies from the analysis or should we give them a very low weight to compute the average? In economic terms, maybe neither of these options seems a reasonable decision because Amazon and Google are third and fifth in the list of the 10 largest components of the S&P 500. An alternative way would be to keep the two companies in the data set and to compute a measure that does not penalize the higher observations as much as the median, harmonic and geometric averages. The adjusted median, the measure proposed in this paper, meets that purpose because while the influence of higher data points is not as high as in the arithmetic average, it nevertheless gives more weight to the higher observations than the median and the other two averages.

The role of an average is to represent a data set meaningfully, and the decision to choose the appropriate average to represent the central tendency of a distribution is an old and yet highly topical matter, in view of what was written by Coggeshall a long time ago (1886, p. 84). His contention was that the mean commonly employed by the economist is not a real quantity at all, but is a quantity assumed as the representative number of others that differ from it to a greater or lesser extent. Its fictitious character renders it possible to choose from among different values, and thus among different methods of finding it. Even in terms of mental images, the word average can lead to different meanings, reinforcing the doubts when we think on it (Kaplan et al. 2010). When asked to compute an average,[1] many practitioners assume the arithmetic mean is what is called for. However, and very often, they are not aware that better alternative approaches are available to capture the central tendency of a distribution (Coggeshall 1886), namely the geometric and harmonic means. Knowing which one to use for your data means understanding their differences. For example, in the case of ratios the choice of averaging method does matter, and sometimes the much less familiar harmonic mean provides a more logical approach to averaging the ratio between two magnitudes (Agrrawal et al. 2010). For square contingency tables, Nakagawa et al. (2020) proposes, as an alternative to the weighted arithmetic mean to represent the degree of departure from the marginal homogeneity, a measure which is expressed as a weighted geometric mean of the diversity index.

The averages, and the central tendency measures in general, continue to be used in most of scientific fields. In Economics and Finance, e.g., Gan et al. (2020),

[1] We use “average” or “mean” interchangeably.

Wellalage and Fernandez (2019) and Cheuk and Vorst (1999). In Education, e.g., Assari et al. (2020). In Mathematics, Statistics and Econometrics, e.g., Maki and Ota (2020), Kolahdouz et al. (2020), Priam (2020), del Barrio et al. (2019) and Gou et al. (2019). In Climate Change, e.g., Lyu et al. (2020). In Complexity Systems, e.g., Wu et al. (2020). In Psychology, e.g., Wenzel and Kubiak (2020). In Logistics, e.g., Choi et al. (2019), just to mention a few. In view of their importance, the main purpose of this paper is to clarify differences regarding averaging methods. The contribution we make is fivefold. First, we show in Sect. 2 that the harmonic mean is equivalent to a weighted arithmetic average, where the weights are inversely proportional to the original values, and they are computed in such a way that the contribution of each value to the final average is exactly the same. Thus, the weights compensate the original values to make the contribution of each value equal to the final average. We also generalize the new formula, taking the harmonic and arithmetic averages as particular cases.

Second, different central tendency measures give different interpretations of the center of a distribution. In Sect. 2.2 we show that the median is the center of the distribution in terms of the observations counted, no matter the value of the observations. The arithmetic mean is such that the absolute deviations to the right of it are compensated by the absolute deviations on its left. So, the center is defined in terms of the absolute deviations (or distances) between each value and the arithmetic average. The geometric average defines the center of the distribution in terms of the compounding relative (percentage) deviations. The negative deviations in relative terms are balanced by the positive ones. Finally, the harmonic mean defines the center of the distribution in order that the weighted deviations on its left compensate the weighted deviations on its right, and the weights are inversely proportional to the original values.

Third, the traditional central tendency measures do not properly handle outlying observation. The arithmetic average is dominated by outlying observations. The insensitivity of the geometric and harmonic averages to outliers can obscure large values that may be consequential. Finally, the median does not use all available data and can be misleading with regard to distributions with a long tail because it discards so much information. Due to the drawbacks of traditional measures, we propose a modified one—the adjusted median—in Sect. 3. The adjusted median originates an intermediate value between the median and the arithmetic average, giving more weight to the higher observations than the median and the other two averages. However, the contribution of each value to the final result is not exactly the same as in the harmonic average. Monte Carlo simulation shows the intermediate position of the adjusted median. We also propose a simple measure of skewness, taking the median as the reference.

Fourth, to compute and compare the measures based on real economic data, we use the daily stock price of 56 companies listed in the "Information Technology" sector of the S&P 500. The data set includes Amazon, Google, Microsoft and Apple, among many others. We show that the adjusted median represents the center of the daily stock price distribution without excluding or giving a very low weight to the outlying observations. Finally, we provide the R code to perform the calculations arising in this study.

The outline of the paper is as follows. First, we review the traditional averaging methods suggested in statistics textbooks, and used by academic researchers and practitioners. We also discuss some particularities of the means leading to a better interpretation and understanding. A new measure, the adjusted median, is proposed in Sect. 3. Monte Carlo simulation studies are performed in Sect. 4 to show the location of this measure and a real data example is also considered in Sect. 5. Finally, we present our concluding remarks.

2 Revisiting the Central Tendency Measures

The classic methods of averaging data are the three Pythagorean means: the familiar arithmetic mean, the geometric mean (the *n*th root of the product of the numbers) and the harmonic mean (the reciprocal of the arithmetic mean of the reciprocals of the numbers).

The arithmetic mean, or simply the mean or average, is the sum of all the X values divided by the number of observations (n):

$$\bar{X_A} = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{\sum_{i=1}^{n} x_i}{n}, \quad i = 1, 2, \ldots, n, \tag{1}$$

where x_i represents the different values assumed by the X variable. See, for example, Chen (1995) for statistical inference about the arithmetic average in the case of positively skewed distributions.

The geometric mean is another type of average, pointing to the central tendency of a set of numbers, and is defined as the root of order n of the product of values x_i:

$$\bar{X_G} = \sqrt[n]{\prod_{i=1}^{n} x_i} = \sqrt[n]{x_1 \times x_2 \times \ldots \times x_n} = (x_1 \times \ldots \times x_n)^{\frac{1}{n}}, \tag{2}$$

where the capital letter Π represents a series of multiplications (or products). As Galton (1897) suggested in one of the earliest papers on geometric average, the distribution of $\bar{X_G}$ will approach normality as n increases, for all parent distributions to which the central limit theorem applies. Thus, the distribution of $\bar{X_G}$ will approach the log-normal form, even though the parent distribution of X may not be log-normal (Alf and Grossberg 1979). The geometric mean applies only to numbers of the same sign in order to avoid the nth root of a negative number when n is even, which is not defined in the real numbers set. In general only positive numbers are allowed (Excel, for example).

Let $\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}$ be the reciprocals of the given set of observations. The equally weighted harmonic mean is expressed as the reciprocal of the arithmetic mean of the reciprocals:

$$\bar{X}_H = \left(\frac{\sum_{i=1}^{n}\frac{1}{x_i}}{n}\right)^{-1} = \frac{n}{\sum_{i=1}^{n}\frac{1}{x_i}}. \tag{3}$$

If different weights $\delta_1, \delta_2, \ldots, \delta_n$ are assigned to the x_i observations, the weighted harmonic mean is defined by:

$$\bar{X}_H = \frac{\sum_{i=1}^{n}\delta_i}{\sum_{i=1}^{n}\frac{\delta_i}{x_i}}. \tag{4}$$

The relation between the three means is: $\bar{X}_H \leq \bar{X}_G \leq \bar{X}_A$ (for strictly positive values). Thus, the arithmetic average is the largest one.[2]

What we propose in this paper is a measure between the arithmetic average and the median that better accommodates the outlying observations (see Sect. 3).

2.1 Contributions to the Final Average

As

$$\bar{X}_H = \sum_{i=1}^{n} x_i \frac{w_i}{\sum_{i=1}^{n} w_i} = \frac{n}{\sum_{i=1}^{n}\frac{1}{x_i}}, \tag{5}$$

where $w_i = \frac{\sum_{i=1}^{n} x_i}{x_i}$, the harmonic mean is equivalent to a weighted arithmetic average, where the weights are inversely proportional to the original values and are computed in such a way that the contribution of each value (x_i) to the final average is exactly the same[3]:

$$x_1 \frac{w_1}{\sum_{i=1}^{n} w_i} = x_2 \frac{w_2}{\sum_{i=1}^{n} w_i} = \cdots = x_i \underbrace{\frac{w_i}{\sum_{i=1}^{n} w_i}}_{p_i} = \cdots = x_n \frac{w_n}{\sum_{i=1}^{n} w_i} = \frac{1}{\sum_{i=1}^{n}\frac{1}{x_i}}. \tag{6}$$

If x_m is the minimum value and its (highest) weight is p_m, the weight (smaller) of the other values x_i is given by $p_i = \frac{x_m}{x_i} p_m$: the weight is inversely proportional to the values, where the constant contribution of each value to the harmonic average is given by $k = x_i \times p_i$.

Thus, all data points, in spite of their value, make the same contribution to the harmonic average, giving more weight to the smallest observations in the data set. However, due to its insensitivity to outliers, the harmonic mean (like the geometric mean, as we will see next) can obscure large values that may be consequential.

To compute the harmonic mean by Eq. (5) we can use the R function whose code is presented as follows:

```
HarmonicMean <- function(x){
weigs1 = sum(x)/x
```

[2] See "Appendix B" for demonstration.

[3] See "Appendix A" for demonstrations.

```
weigs = weigs1/sum(weigs1)
HarmonicMean <- t(x)% * %weigs
return(HarmonicMean)}
```

To compute the harmonic mean by using the standard formula we can use the R function:

```
hm_mean < - function(a){length(a)/(sum(1/a))}
```

Equation (5) can be generalized to:

$$\bar{X}_{GA} = \sum_{i=1}^{n} x_i \frac{w_i^k}{\sum_{i=1}^{n} w_i^k}, \tag{7}$$

where k is a real number. If $k = 1$, $\bar{X}_{GA}$ is the harmonic average; if $k = 0$, $\bar{X}_{GA}$ is the arithmetic average. The geometric average is obtained for a particular value of k in the range $0 < k < 1$, which is not always constant.

In the (simple) arithmetic average what is constant is the weight associated with each data point:

$$\bar{X}_A = \sum_{i=1}^{n} \frac{1}{n} x_i = \frac{1}{n} x_1 + \frac{1}{n} x_2 + \ldots + \frac{1}{n} x_n. \tag{8}$$

Therefore, if the values are different, the contribution of each $(c_i = \frac{1}{n} x_i)$ to the arithmetic average is not constant and higher values will make a greater contribution, which is directly proportional to the value; this is the reason why the arithmetic average is influenced by outlying observations (Lyu et al. 2020). The constant of proportionality is $k = \frac{c_i}{x_i}$ and $c_i = kx_i = \frac{1}{n} x_i$.

The simplest way to reduce the importance of extreme points is to use the harmonic or geometric averages as an alternative to the arithmetic average when the data set includes outlying observations. The geometric average is the root of order n of the product of x_i's and the contribution of each value to the geometric average is neither constant nor directly proportional to the respective value.

By taking the natural log of both sides of Eq. (2) we get:

$$\ln(\bar{X}_G) = \frac{1}{n} \sum_{i=1}^{n} \ln(x_i) = \frac{1}{n} \ln(x_1) + \frac{1}{n} \ln(x_2) + \cdots + \frac{1}{n} \ln(x_n). \tag{9}$$

By taking the anti-logarithm of both sides:

$$\bar{X}_G = \exp\left[\frac{1}{n} \sum_{i=1}^{n} \ln(x_i)\right] = x_1^{\frac{1}{n}} x_2^{\frac{1}{n}} \cdots x_n^{\frac{1}{n}}. \tag{10}$$

The R code of two functions to compute the geometric mean is:

```
gm_mean <- function(a){ prod(a)^(1/length(a)) }or
gm_meanlog <- function(a){
```

```
len <- length(a)
gm_meanlog <- exp(1/len * sum(log(a)))
return(gm_meanlog) }
```

Based on Jensen inequality,[4] we conclude that the value of the arithmetic average is at least the value of the geometric average: $\bar{X}_A \geq \bar{X}_G$ (where the equality holds only if all the x_i are equal). The contribution of each data point to the geometric average is less than proportional to the x_i values, reducing the impact of outlying observations on the final value. Thus, the contribution of each data point to the geometric average is neither constant, as in the harmonic mean, nor directly proportional to the values, as in the arithmetic average, resulting in an intermediate value between harmonic and arithmetic averages.

2.2 Different Meanings of the Center

Choosing the appropriate mean to represent the central tendency of a distribution is an old and yet highly topical matter, in view of what was written by Coggeshall a long time ago (1886, p. 84) where he stated that the mean commonly employed by the economist is not a real quantity at all, but is a quantity assumed as the representative number of others that more or less differ from it. Its fictitious character renders it possible to choose from among different values, and thus among different methods of finding it. The same conclusion holds for the median when the number of observations is even.

Let us consider first the median to represent the central tendency of a distribution. By ranking the observations from the lowest to the highest, the ranks associated with each observation are: $r_1, r_2, \ldots, r_n$ where r_1 and r_n represent the minimum and the maximum, respectively. The median $(\bar{X}_M)$ corresponds to the middle point in terms of counted observations (absolute frequencies) in spite of the respective value: $r_M = \frac{r_1 + r_n}{2}$ and the following property holds for the median:

$$\sum_{i=1}^{n} (r_i - r_M) = 0, \tag{11}$$

where r_i represents the ranks of ordered observations and r_M is the rank corresponding to the median. The sum of the differences between the ranks, and the rank corresponding to the median, is zero.

The arithmetic average $(\bar{X}_A)$ defines the center of the distribution in terms of the observations' value:

$$\sum_{i=1}^{n} (x_i - \bar{X}_A) = 0, \tag{12}$$

where x_i represents the value of each observation in the data set.

4 See "Appendix B" for demonstrations.

The geometric average ($\bar{X}_G$) defines the center of the distribution in terms of the log percentage deviations (Trönqvist et al. 1985) around $\bar{X}_G$ and the next property holds for this average:

$$\sum_{i=1}^{n} \ln\left(\frac{x_i}{\bar{X}_G}\right) = 0, \tag{13}$$

or the sum of the differences between the log of x_i and the log of the geometric average is zero:

$$\sum_{i=1}^{n} [\ln(x_i) - \ln(\bar{X}_G)] = 0, \tag{14}$$

where $\ln(x_i) - \ln(\bar{X}_G)$, when multiplied by 100, is the log percentage deviation between each observation and the geometric average. Thus, the geometric average is the value that balances the negative log percentage deviations with the positive ones.

The harmonic average ($\bar{X}_H$) defines the center of the distribution in terms of the absolute deviations weighted by $\frac{w_i}{\sum_{i=1}^{n} w_i}$ (see Eq. (5)):

$$\sum_{i=1}^{n} (x_i - \bar{X}_H) \frac{w_i}{\sum_{i=1}^{n} w_i} = 0. \tag{15}$$

Thus, the harmonic mean defines the center of the distribution such that the weighted deviations on its left compensate the weighted deviations on its right. The weights are inversely proportional to the original values.[5]

Between the median, harmonic, geometric and arithmetic averages, which measure should be used to represent the central tendency of a distribution? We very often use the arithmetic mean and the median, occasionally the geometric mean, and very rarely the harmonic mean. However, there is nothing to actually prevent us from using whatever measure we would like; we have to choose the one most suitable for each particular situation.

For a sequence of independent (non-outlying) observations, the arithmetic mean is the preferred measure to represent the central tendency because all the available data is used to compute the respective value, while to the median contributes only half of the data. This is the case, for example, with regard to the financial incomes of different companies. If a company happens to perform poorly, the chance of other companies doing better isn't affected. In general, statisticians use arithmetic means to represent independent data with no significant outliers. However, when it comes to annual investment returns, the numbers are not independent from each other. If you lose a ton of money one year, you have much less capital to generate returns during the following years. So, we cannot say that yearly returns are independent, and the geometric average is more suitable to represent the center. Thus, the geometric mean is more appropriate for data series that exhibit serial correlation.

[5] See "Appendix C" for simple applications of the results of this subsection.

When at least one of the observations is zero the arithmetic mean is still more appropriate to average the observations because the geometric mean is zero and the harmonic mean cannot be computed as the reciprocals of the original values are undefined.

In the presence of outlying observations, the harmonic mean of a list of numbers tends strongly toward the smaller elements of the list; it tends (compared to the arithmetic mean) to mitigate the impact of large outliers and to aggravate the impact of the smallest ones. When you look at the results of arithmetic and geometric means calculations, we notice that the effect of outliers is greatly dampened in the geometric mean.

As the arithmetic average is not a robust statistic, meaning that it is influenced by extreme observations, and the harmonic and geometric averages tend to mitigate the impact of large observations, the median is used more often to represent the central tendency because it is robust towards outlying observations. However, the median is computed based on the ranks and not on the value of the observations. Thus, in the next section we propose a new measure that incorporates all data values in its calculation and which, compared to traditional measures, provides an alternative description of the central tendency when there are outlying observations.

3 The Adjusted Median

We have already pointed out the drawbacks of the three averages. The arithmetic average is dominated by outlying observations (Matthews 2004). And the insensitivity of the geometric and harmonic averages to outliers can obscure large values that may be consequential. The median does not use all the available data and can be misleading in distributions with a long tail because it discards so much information.

Due to the drawbacks of traditional central tendency measures in the presence of outlying observations, we propose the adjusted Median: $\bar{X}_{aM}$ (aMED), for its capacity to provide an intermediate value able to take into account the extreme observations, between the lowest (harmonic, geometric and median) and the highest (arithmetic) averages. Let S_L and S_R be the sum of x_i deviations to the left and to the right of the median:

$$S_L = \sum_{x_i < \bar{X}_M}^{n_1} (\bar{X}_M - x_i) \quad \text{and } S_R = \sum_{x_i > \bar{X}_M}^{n_2} (x_i - \bar{X}_M). \tag{16}$$

The adjusted Median $(\bar{X}_{aM})$ is defined as:

$$\bar{X}_{aM} = \bar{X}_M + \underbrace{\frac{S_R - S_L}{S_R + S_L}}_{sk} |\bar{X}_A - \bar{X}_M|, \tag{17}$$

where $\bar{X}_M < \bar{X}_{aM} < \bar{X}_A$, when the distribution is asymmetric positive and the arithmetic average is higher than the median: $\bar{X}_A > \bar{X}_M$; $\bar{X}_A < \bar{X}_{aM} < \bar{X}_M$, when the distribution is asymmetric negative and the arithmetic average is lower than the median: $\bar{X}_A < \bar{X}_M$. If the distribution is symmetric, $S_R = S_L$ and $\bar{X}_{aM} = \bar{X}_M = \bar{X}_A$.

The ratio

$$sk = \frac{S_R - S_L}{S_R + S_L} \tag{18}$$

can also be used as a simple measure of skewness (taking the median as the reference) pointing to an asymmetric positive, asymmetric negative or symmetric distribution, when its value is positive, negative or zero, respectively. The advantage of this measure is that it always ranges between -1 and $+1$.

Thus, the adjusted median causes the median to shift towards the arithmetic average and its location depends on the balance between the two sides of the median (not in terms of frequencies, but in terms of values, represented by the sums S_L and S_R). This formula uses a linear interpolation to estimate the adjusted median and it follows the usual method to calculate the mode in continuous grouped data (Basu and DasGupta 1997). $\bar{X}_{aM}$ originates an intermediate[6] value between the median and arithmetic average, bringing more weight to the extreme observations, when compared to the median, harmonic and geometric averages, and less weight when compared to the arithmetic average. Thus, outlying observations are still influential in the final result, but not as much as in the arithmetic average.

The code of the R function to compute the adjusted median is shown as follows:

```
adj_median <- function(x){
med1 <- median(x)
sumleft <- sum(med1-x[x<med1])
sumright <- sum(x[x>med1]-med1)
tot < - sumleft + sumright
sk <- (sumright-sumleft)/tot
diff1 <- abs(mean(x)-median(x))
diff2 < - sk *diff1
if(mean(x)< median(x)) {
aMED < - median(x)-diff2 } else {
aMED <- median(x)+diff2 }
return(list(aMED,sk)) }
```

The adjusted median ($\bar{X}_{aM}$) does not penalize the higher data points as much as the median, harmonic and geometric means, and it gives less weight to those observations, when compared to the arithmetic average. Thus, in the harmonic and geometric averages (especially the first) the higher data points are penalized excessively, and the resulting mean is too small to represent the outlying observations. On the other hand, the arithmetic average exacerbates the effect of those observations, giving rise to a mean that is generally too high (or too low). An adjusted median, therefore, can be an attractive way to deal with outlying observations: these values still contribute to $\bar{X}_{aM}$, but the weight is lower when

[6] See Sect. 4 for details.

compared to the arithmetic average and higher when compared to the median, harmonic and geometric means.

What are the differences of the adjusted median when compared to the traditional means and median? The adjusted median provides that higher observations remain in the data set without giving as much weight as in the arithmetic average, or too small a weight as in the other three central tendency measures. Thus, it is an alternative way to deal with outlying observations. Furthermore, there are no constraints to computing the adjusted median even in the context of variables that can take non-positive values, which is a shortcoming of geometric and harmonic averages, as previously pointed out.

4 Simulation Study

A simulation study was conducted to confirm the intermediate position of the adjusted median among the central tendency measures. We simulate 10,000 Monte Carlo samples of different sizes: 10, 20, 30, 40, 50 and 100, from various strictly positive distributions ($x_i > 0$): beta: B(3,3) and B(1,10), lognormal: LN(0,1) and LN(2,1), weibull: WB(2,1) and WB(5,2), gamma: G(1,6) and G(2,5), exponential: EXP(1) and EXP(5), and chi-squared with 1 and 10 degrees of freedom: χ_1^2 and χ_{10}^2 (see Tables 1, 2 for results). The simulation routines have been programmed in R and are available on request.

The columns "NOUT" contain the mean of each measure (Harmonic, Geometric, Median, adjusted Median and Arithmetic) computed based on the 10,000 Monte Carlo samples. Thus, for each sample we compute all the five central tendency measures, resulting in 10,000 different values per measure. Next, the mean of each measure is computed and is shown in columns "NOUT". After simulating the data, we manually introduce one, two, three, four, five and ten severe outliers in the samples with sizes 10, 20, 30, 40, 50 and 100, respectively, randomly replacing the original observations with the value: $x_{SO} = Q_3 + 3 \times IQR$, the usual threshold for severe outliers, where Q_3 and IQR are the third quartile and the inter-quartile range, respectively. The mean of the measures computed "with outliers" is shown in the columns "OUT".

As can be seen in Tables 1 and 2, the value of the adjusted median is always between the geometric/median and the arithmetic average, confirming its intermediate position. Thus, being closer to the arithmetic average also accommodates the outlying observations, without exacerbating its effect on the final result. The distributions, with the exception of the symmetric B(3,3), are all asymmetric positive and become even more asymmetrical when severe upper outliers are introduced. These conclusions are based on the "SK" estimates resulting from the proposed measure in (18), which are almost zero for B(3,3) and positive for the remaining distributions. The value of the estimates also increases with outlying observations, pointing to an even longer right tail.

To confirm that outlying observations do not affect the median, harmonic and geometric means as they do with regard to the arithmetic average, we compare the simulation results without ("NOUT") and with ("OUT") outliers. As can be seen,

Table 1 Central tendency measures, adjusted median and skewness

		HA		GA		MED		aMED		AA		SK	
Dist.	*n*	NOUT	OUT	NOUT	OUT	NOUT	OUT	NOUT	OUT	NOUT	OUT	NOUT	OUT
B(3,3)	10	0.418	0.449	0.462	0.512	0.499	0.527	0.500	0.548	0.500	0.583	0.001	0.261
	20	0.410	0.441	0.459	0.513	0.500	0.530	0.500	0.551	0.500	0.591	− 0.001	0.272
	30	0.408	0.439	0.459	0.514	0.501	0.530	0.501	0.551	0.500	0.594	− 0.002	0.277
	40	0.405	0.437	0.458	0.513	0.500	0.529	0.500	0.550	0.500	0.594	0.001	0.281
	50	0.404	0.436	0.458	0.514	0.500	0.529	0.500	0.550	0.500	0.596	0.001	0.284
	100	0.402	0.441	0.457	0.526	0.500	0.537	0.500	0.563	0.500	0.616	− 0.001	0.317
B(1,10)	10	0.032	0.036	0.057	0.069	0.071	0.082	0.081	0.103	0.091	0.121	0.329	0.458
	20	0.026	0.029	0.056	0.068	0.069	0.080	0.080	0.103	0.091	0.123	0.364	0.490
	30	0.023	0.026	0.055	0.067	0.068	0.079	0.078	0.103	0.091	0.123	0.375	0.502
	40	0.021	0.024	0.054	0.067	0.068	0.079	0.078	0.103	0.091	0.124	0.381	0.506
	50	0.020	0.023	0.054	0.067	0.068	0.079	0.078	0.103	0.091	0.124	0.381	0.508
	100	0.017	0.020	0.054	0.069	0.067	0.081	0.077	0.108	0.091	0.131	0.389	0.526
LN(0,1)	10	0.699	0.775	1.050	1.251	1.087	1.264	1.446	1.790	1.648	2.094	0.464	0.539
	20	0.652	0.722	1.023	1.226	1.038	1.202	1.416	1.781	1.651	2.109	0.521	0.586
	30	0.639	0.704	1.019	1.220	1.028	1.184	1.409	1.779	1.657	2.117	0.541	0.602
	40	0.632	0.697	1.014	1.216	1.022	1.177	1.399	1.770	1.652	2.113	0.547	0.606
	50	0.624	0.688	1.008	1.209	1.015	1.169	1.389	1.763	1.645	2.107	0.552	0.610
	100	0.619	0.695	1.007	1.255	1.010	1.201	1.385	1.839	1.652	2.209	0.565	0.623
LN(2,1)	10	5.209	5.780	7.810	9.300	8.072	9.378	10.708	13.237	12.194	15.491	0.462	0.537
	20	4.828	5.337	7.566	9.049	7.681	8.875	10.438	13.109	12.163	15.519	0.519	0.584
	30	4.721	5.209	7.513	9.001	7.571	8.738	10.359	13.087	12.186	15.577	0.541	0.600
	40	4.666	5.144	7.478	8.967	7.546	8.699	10.287	13.041	12.155	15.569	0.545	0.605
	50	4.635	5.105	7.469	8.958	7.527	8.675	10.279	13.043	12.180	15.596	0.551	0.609

Table 1 continued

Dist.	*n*	HA		GA		MED		aMED		AA		SK	
		NOUT	OUT	NOUT	OUT	NOUT	OUT	NOUT	OUT	NOUT	OUT	NOUT	OUT
	100	4.556	5.115	7.424	9.246	7.450	8.852	10.220	13.571	12.189	16.301	0.565	0.623
WB(2,1)	10	0.635	0.691	0.765	0.870	0.842	0.910	0.861	0.983	0.887	1.079	0.121	0.330
	20	0.608	0.663	0.758	0.867	0.838	0.906	0.854	0.983	0.888	1.093	0.135	0.352
	30	0.595	0.648	0.753	0.863	0.833	0.900	0.848	0.978	0.884	1.095	0.140	0.360
	40	0.590	0.643	0.752	0.864	0.834	0.902	0.848	0.979	0.885	1.099	0.139	0.361
	50	0.586	0.639	0.752	0.864	0.834	0.902	0.846	0.979	0.886	1.101	0.141	0.363
	100	0.576	0.639	0.750	0.888	0.832	0.916	0.843	1.010	0.886	1.148	0.144	0.393
WB(5,2)	10	1.736	1.828	1.790	1.916	1.857	1.914	1.850	1.950	1.838	2.014	0.055	0.219
	20	1.726	1.823	1.785	1.919	1.856	1.914	1.850	1.949	1.836	2.024	0.061	0.230
	30	1.724	1.822	1.785	1.923	1.858	1.917	1.853	1.950	1.837	2.031	0.061	0.234
	40	1.723	1.822	1.784	1.923	1.859	1.917	1.853	1.950	1.837	2.033	0.064	0.235
	50	1.723	1.822	1.784	1.924	1.859	1.918	1.854	1.950	1.837	2.034	0.066	0.236
	100	1.720	1.842	1.783	1.954	1.858	1.931	1.855	1.973	1.837	2.076	0.064	0.275

We simulate 10,000 Monte Carlo samples of different sizes: 10, 20, 30, 40, 50 and 100 from distributions: beta: B(3,3) and B(1,10), lognormal: LN(0,1) and LN(2,1), weibull: WB(2,1) and WB(5,2). The values in the table represent the mean of each measure resulting from the 10,000 samples. "NOUT" and "OUT" represent the simulation results without and with outliers, respectively. "SK" is the skewness measure defined in (18). *HA* harmonic, *GA* geometric, *MED* median, *aMED* adjusted median, *AA* arithmetic average

Table 2 Central tendency measures, adjusted median and skewness

		HA		GA		MED		aMED		AA		SK	
Dist.	*n*	NOUT	OUT	NOUT	OUT	NOUT	OUT	NOUT	OUT	NOUT	OUT	NOUT	OUT
G(1,6)	10	0.854	0.916	0.927	1.026	0.955	1.011	0.974	1.076	1.000	1.160	0.140	0.341
	20	0.844	0.906	0.923	1.027	0.950	1.006	0.967	1.076	1.001	1.173	0.159	0.365
	30	0.840	0.902	0.920	1.026	0.947	1.004	0.963	1.075	1.000	1.175	0.164	0.371
	40	0.839	0.902	0.921	1.028	0.948	1.005	0.963	1.075	1.001	1.179	0.164	0.374
	50	0.837	0.901	0.920	1.028	0.947	1.003	0.961	1.075	1.001	1.180	0.168	0.378
	100	0.835	0.912	0.918	1.050	0.946	1.014	0.957	1.098	0.999	1.216	0.170	0.405
G(2,5)	10	1.655	1.778	1.828	2.035	1.896	2.017	1.943	2.167	2.004	2.352	0.155	0.350
	20	1.622	1.747	1.811	2.030	1.876	2.000	1.919	2.160	1.997	2.371	0.174	0.373
	30	1.614	1.740	1.808	2.030	1.873	1.996	1.912	2.158	1.997	2.380	0.179	0.381
	40	1.614	1.740	1.809	2.033	1.873	1.995	1.909	2.157	1.999	2.384	0.182	0.385
	50	1.610	1.736	1.807	2.033	1.871	1.995	1.906	2.158	2.000	2.389	0.185	0.387
	100	1.606	1.760	1.806	2.082	1.870	2.020	1.900	2.212	2.000	2.471	0.187	0.415
EXP(1)	10	0.328	0.375	0.609	0.740	0.748	0.873	0.887	1.129	1.001	1.329	0.366	0.480
	20	0.260	0.295	0.585	0.716	0.721	0.840	0.864	1.127	1.002	1.350	0.407	0.517
	30	0.233	0.265	0.577	0.710	0.710	0.831	0.851	1.123	1.000	1.354	0.420	0.527
	40	0.218	0.247	0.574	0.706	0.706	0.825	0.846	1.122	1.000	1.358	0.426	0.533
	50	0.204	0.232	0.570	0.703	0.703	0.822	0.843	1.121	1.000	1.360	0.430	0.536
	100	0.175	0.204	0.567	0.730	0.699	0.848	0.837	1.178	1.001	1.438	0.436	0.551
EXP(5)	10	1.636	1.875	3.037	3.698	3.717	4.351	4.422	5.644	5.008	6.648	0.373	0.484
	20	1.291	1.470	2.912	3.572	3.589	4.189	4.303	5.627	5.002	6.743	0.409	0.518
	30	1.175	1.333	2.898	3.563	3.566	4.169	4.277	5.647	5.025	6.810	0.421	0.528
	40	1.080	1.226	2.872	3.533	3.534	4.128	4.236	5.615	5.006	6.797	0.426	0.533
	50	1.020	1.157	2.856	3.520	3.524	4.116	4.219	5.607	5.005	6.803	0.429	0.536

Table 2 continued

Dist.	n	HA NOUT	HA OUT	GA NOUT	GA OUT	MED NOUT	MED OUT	aMED NOUT	aMED OUT	AA NOUT	AA OUT	SK NOUT	SK OUT
	100	0.875	1.013	2.832	3.644	3.494	4.232	4.187	5.895	5.009	7.197	0.437	0.552
χ_1^2	10	0.100	0.122	0.352	0.457	0.547	0.697	0.849	1.139	1.001	1.373	0.538	0.586
	20	0.050	0.061	0.314	0.413	0.499	0.637	0.825	1.132	0.998	1.380	0.590	0.630
	30	0.033	0.040	0.301	0.398	0.483	0.618	0.815	1.128	0.995	1.382	0.606	0.643
	40	0.025	0.031	0.299	0.396	0.479	0.615	0.814	1.135	1.000	1.394	0.613	0.649
	50	0.020	0.024	0.294	0.390	0.473	0.605	0.811	1.132	0.999	1.392	0.618	0.654
	100	0.010	0.013	0.289	0.406	0.466	0.631	0.807	1.195	1.001	1.477	0.626	0.659
χ_{10}^2	10	13.231	14.122	14.106	15.490	14.431	15.199	14.664	16.045	14.989	17.153	0.129	0.332
	20	13.133	14.050	14.081	15.547	14.402	15.171	14.613	16.072	15.023	17.351	0.144	0.354
	30	13.097	14.015	14.064	15.546	14.390	15.143	14.573	16.042	15.016	17.380	0.146	0.360
	40	13.053	13.979	14.031	15.533	14.352	15.108	14.526	16.017	14.994	17.397	0.150	0.366
	50	13.053	13.983	14.039	15.549	14.372	15.130	14.530	16.027	15.009	17.426	0.148	0.365
	100	13.020	14.158	14.019	15.872	14.344	15.275	14.480	16.360	14.999	17.938	0.153	0.396

We simulate 10,000 Monte Carlo samples of different sizes: 10, 20, 30, 40, 50 and 100 from distributions: gamma: G(1,6) and G(2,5), exponential: EXP(1) and EXP(5), and chi-squared with 1 and 10 degrees of freedom: χ_1^2 and χ_{10}^2. The values in the table represent the mean of each measure resulting from the 10,000 samples. "NOUT" and "OUT" represent the simulation results without and with outliers, respectively. "SK" is the skewness measure defined in (18). *HA* harmonic, *GA* geometric, *MED* median, *aMED* adjusted median, *AA* arithmetic average whereas the harmonic mean has a small increase (or remains more or less constant), and the median and geometric averages show a moderate increase, the arithmetic average increases sharply. Thus, the outliers strongly affect the arithmetic mean, modestly affect the median and geometric mean, and have a small impact on the harmonic mean.

The simulation results also indicate that the adjusted median's increase is higher than those of the median, harmonic and geometric means, giving more weight to the outlying data points. At the same time, it is smaller than that of the arithmetic mean, confirming that the adjusted median can be used to represent the central tendency of a distribution as an alternative to the median, geometric and arithmetic means in the presence of outlying observations.

5 S&P 500 Information Technology

To compute and compare the measures based on real economic data, we used the daily stock price of 56 companies listed in the sector "Information Technology" of Standard and Poor's 500 (S&P 500([7])). The data set includes the daily stock prices of Microsoft, Apple, Amazon, Google, among many other companies.

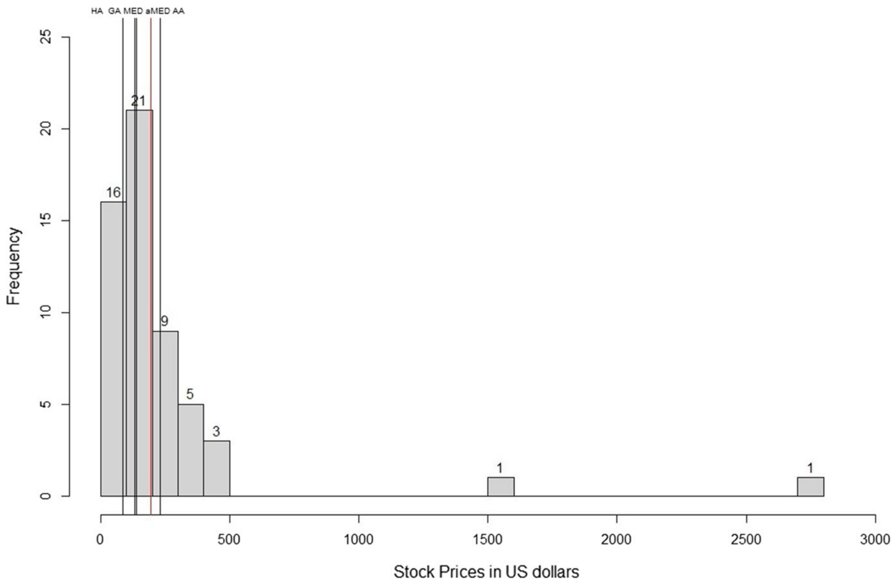
5.1 The Adjusted Median in the Central Tendency

Figures 1 and 2 show the histograms of the stock prices including and excluding two outlying observations (the two companies with the highest price are Amazon and Google). The vertical lines correspond to the value of the means, median and adjusted median and their position is determined by its value (see Table 3). The number of companies in each interval is shown at the top of each column (for example, there are 21 companies with price between 100 and 200).

In both cases, the center seems to be in the interval 100–200. When Amazon and Google are included, neither the harmonic nor the arithmetic averages seem appropriate to represent the central tendency of the distribution. From the other three measures (geometric, median and adjusted median) the latter deviates to the upper limit of the interval (200) reflecting also the extreme observations to the right of the distribution. When the outliers are removed, the adjusted median is very close to the median, but is higher than the harmonic and geometric averages, reflecting the positive asymmetry that still remains. Therefore, it appears to remain appropriate to represent the center of the distribution. As the value of the adjusted median is not influenced as much as the arithmetic average, the outlying observations can still contribute to representing the center of the distribution without it deviating too much from the other central tendency measures.

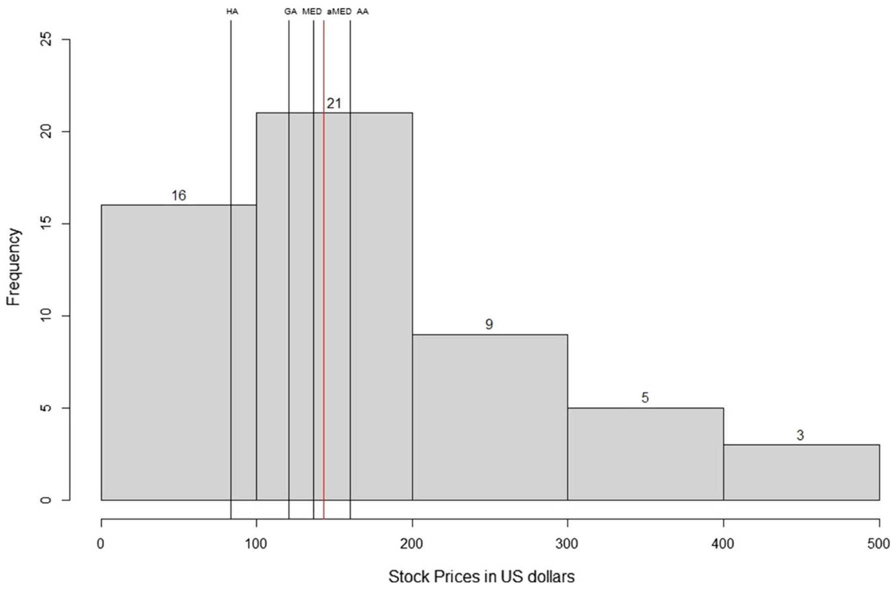
Table 3 presents descriptive statistics for companies of the S&P 500 Information Technology sector. The asymmetry and kurtosis is, as expected, substantially reduced when the two extreme prices are removed. The harmonic mean and the median are relatively constant, demonstrating their resilience to extreme

[7] The data source is: https://www.tradingview.com. Prices refer to June 30, 2020.



HA: Harmonic, GA: Geometric, AA: Arithmetic averages, MED: Median, aMED: adjusted Median

Fig. 1 S&P 500 Information Technology with AMAZON and GOOGLE. *HA* harmonic, *GA* geometric, *AA* arithmetic averages, *MED* median, *aMED* adjusted median



HA: Harmonic, GA: Geometric, AA: Arithmetic averages, MED: Median, aMED: adjusted Median

Fig. 2 S&P 500 Information Technology excluding AMAZON and GOOGLE. *HA* harmonic, *GA* geometric, *AA* arithmetic averages, *MED* median, *aMED* adjusted median

Table 3 Descriptive statistics of daily stock price

Statistic	All	No outliers
Harmonic Average (HA)	86.53	83.57
Geometric Average (GA)	133.30	120.45
Median (MED)	137.46	136.34
adjusted Median (aMED)	194.73	143.09
Arithmetic Average (AA)	231.19	160.24
Minimum	16.35	16.35
Maximum	2758.82	468.87
Skewness	5.00	1.11
Kurtosis	29.72	3.61
# Companies	56	54

observations. The reduction in the geometric mean is not as high as in the adjusted median and arithmetic average. With regard to the adjusted median, it lies between the median and arithmetic average, giving more weight (when compared to the median, geometric and harmonic means) to the higher stock prices (as preciously seen, the harmonic mean gives the least weight to higher data points). As it is still far from the arithmetic mean (especially when Amazon and Google are included), the adjusted median is less influenced by outlying observations when compared to the arithmetic mean.

The traditional ways to deal with outliers is to remove them or to compute central tendency measures robust to outliers. Should we exclude those two companies or should we give them a very low weight when we compute the average of technology stock prices? In economic terms, maybe neither would seem to be a reasonable decision because Amazon and Google are third and fifth in the list of the 10 largest components of the S&P 500. Thus, an alternative way could be to keep the two companies in the data set and compute a measure that does not penalize the higher observations as much as the median, harmonic and geometric means. The adjusted median fits this purpose because although the influence of higher data points is not as high as in the arithmetic average, it does, nevertheless, give more weight to the higher observations than the other three measures.

5.2 Bootstrapping Confidence Intervals

The bootstrap sample comprises the 56 companies listed in the sector "Information Technology" of Standard and Poor's 500. The data set includes the daily stock prices of Microsoft, Apple, Amazon, Google, among many other companies. The statistics of interest are the harmonic, geometric and arithmetic averages, median and adjusted median, and "sk", the simple skewness statistic defined in (18). The bootstrap is used to compute a 95% confidence interval for each measure. The R code to run the bootstrap is presented as follows:

```
bootfunc < - function(data, indices){
dt < - data[indices,]
```

```
c( hm_mean(dt[, 1]),
gm_meanlog(dt[, 1]),
median(dt[, 1]),
adj_median(dt[, 1])[[1]],
mean(dt[, 1]),
adj_median(dt[, 1])[[2]] ) }
data1 < - data.frame(DataFile)
library(boot)
myBootstrap <- boot(data1, bootfunc, R=10000)
myBootstrap
boot.ci(myBootstrap, index=1)
boot.ci(myBootstrap, index=2)
boot.ci(myBootstrap, index=3)
boot.ci(myBootstrap, index=4)
boot.ci(myBootstrap, index=5)
boot.ci(myBootstrap, index=6)
```

The results are shown in Table 4. As can be seen, the limits of the confidence intervals for "sk", are all positive, with the exception of the one resulting from the Accelerated bias-corrected method, where the lower limit is slightly negative. Thus, the distribution of stock prices is asymmetric positive and the estimates for "sk" also increase with the inclusion of Amazon and Google prices, the two upper outlier observations, pointing to an even longer right tail (see Figs. 1, 2).

We can also confirm that the two outlying observations do not affect the median, harmonic and geometric means in the way they affect the arithmetic average. Next, we compare the results without and with the two largest prices. Whereas the limits of the confidence interval for the harmonic mean and median have a small increase, and the ones for the geometric mean increase moderately, the limits for the arithmetic average increase sharply, especially the upper limit. Thus, the two extreme prices strongly affect the arithmetic mean, modestly affect the geometric mean, and have a small impact on the harmonic mean and the median.

The empirical results also confirm that the limits of the confidence interval for the adjusted median (aMED) react more strongly with the inclusion of the two outlying prices: the increase in the limits is higher when compared to those of the median, harmonic and geometric averages, giving more weight to the outlying data points, but it is not as exacerbated as in the arithmetic average. Thus, the intermediate increase also accommodates the outlying data points, confirming the usefulness of the adjusted median as an alternative to the traditional measures, with regard to representing the central tendency of a distribution in the presence of outlying observations.

Table 4 Bootstrap confidence intervals

	HA		GA		MED		aMED		AA		sk	
	Lower	Upper	Lower	Upper	Lower	Upper	Lower	Upper	Lower	Upper	Lower	Upper
Excluding AMAZON and GOOGLE												
Normal	57.88	105.72	93.00	146.00	116.30	162.90	117.90	168.50	128.90	191.00	0.04	0.46
Basic	54.85	102.78	92.10	144.20	123.50	161.50	118.00	168.60	128.50	189.70	0.06	0.47
Percentile	64.36	112.30	96.70	148.90	111.10	149.20	117.60	168.20	130.80	191.90	0.09	0.51
BCα	61.60	107.55	95.70	147.50	110.80	149.00	118.50	169.80	132.20	193.50	− 0.01	0.44
Including AMAZON and GOOGLE												
Normal	59.13	110.17	97.70	166.70	114.10	163.50	99.80	283.00	124.60	338.00	0.35	0.94
Basic	55.61	106.79	94.00	163.00	116.90	162.90	84.00	259.00	111.80	316.40	0.42	1.00
Percentile	66.27	117.45	103.60	172.60	112.00	158.10	130.40	305.50	145.90	350.50	0.22	0.80
BCα	63.46	111.74	104.30	173.70	111.80	158.10	141.30	367.00	160.70	424.80	0.29	0.83

The bootstrap confidence intervals are computed in accordance with the methods: Standard (Normal), Percentile (Percentile), Pivotal or Empirical (Basic) and Accelerated bias-corrected (BCα). See, for example, Gonzãlez-Manteiga et al. (1994), for bootstrap details. *HA* harmonic average, *GA* geometric average, *MED* median, *aMED* adjusted median and *AA* arithmetic average. The estimates are based on 10,000 bootstrap replicates. 10,000 instead of 10000 and the dot to end the sentence

6 Conclusions

The average value of a data set is, possibly, the most common statistical idea encountered in everyday life. When asked to compute an average, many students, as well as practitioners, assume the arithmetic mean is what is called for. However, and very often, they are not aware that better alternative approaches are available to capture the central tendency of a distribution, namely the geometric and harmonic means.

In this paper we revisit the three traditional averages, highlighting their strengths and weaknesses. The arithmetic average is strongly influenced by outlying observations, while the harmonic and geometric means are insensitive to outliers, which can obscure large values that may be consequential. An alternative way is to find a measure that does not penalize the higher observations as much as the harmonic and geometric means.

To overcome the drawbacks of traditional averages, we propose the adjusted median (aMED). The aMED does not penalize the higher data points as much as the median, harmonic and geometric means, and it gives substantially less weight when compared to the arithmetic average. Thus, aMED is an intermediate solution for dealing with outlying observations.

In a Monte Carlo simulation study, we have shown that aMED lies between the median and the arithmetic average, reinforcing our purpose of giving greater weight to higher data points.

To compute and compare the measures based on real economic data, we use the daily stock price of 56 companies listed in the sector "Information Technology" of the S&P 500. The data set includes Amazon, Google, Microsoft and Apple, among many other companies. We show that aMED represents the center of the daily stock price distribution, giving an intermediate weight to the outlying observations, when compared to the traditional central tendency measures.

A Contributions of Each Data Point

See Eq. (5):

$$\bar{X}_H = \sum_{i=1}^{n} x_i \frac{w_i}{\sum_{i=1}^{n} w_i} = \sum_{i=1}^{n} \left[x_i \frac{\frac{\sum_{i=1}^{n} x_i}{x_i}}{\frac{\sum_{i=1}^{n} x_i}{x_1} + \frac{\sum_{i=1}^{n} x_i}{x_2} + \ldots + \frac{\sum_{i=1}^{n} x_i}{x_n}} \right]$$

$$= \sum_{i=1}^{n} \left[\frac{x_i \sum_{i=1}^{n} x_i}{x_i \sum_{i=1}^{n} x_i \sum_{i=1}^{n} \frac{1}{x_i}} \right] =,$$

$$= \sum_{i=1}^{n} \left[\frac{1}{\sum_{i=1}^{n} \frac{1}{x_i}} \right] = \frac{1}{\sum_{i=1}^{n} \frac{1}{x_i}} + \frac{1}{\sum_{i=1}^{n} \frac{1}{x_i}} + \ldots + \frac{1}{\sum_{i=1}^{n} \frac{1}{x_i}} = \frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}},$$

if $x_i \sum_{i=1}^{n} x_i \neq 0$.

See Eq. (7):

$$x_i \frac{w_i}{\sum_{i=1}^n w_i} = x_i \frac{\frac{\sum_{i=1}^n x_i}{x_i}}{\frac{\sum_{i=1}^n x_i}{x_1} + \frac{\sum_{i=1}^n x_i}{x_2} + \ldots + \frac{\sum_{i=1}^n x_i}{x_n}} = \frac{x_i \sum_{i=1}^n x_i}{x_i \sum_{i=1}^n x_i \sum_{i=1}^n \frac{1}{x_i}} = \frac{1}{\sum_{i=1}^n \frac{1}{x_i}},$$

if $x_i \sum_{i=1}^n x_i \neq 0$.

B Averages Inequalities

According to the means definition, see Eqs. (1), (2) and (3), their logarithms are:

$$\ln(\bar{X}) = \ln\left(\frac{1}{n}\sum_{i=1}^n X_i\right), \quad \ln(\bar{X}_G) = \frac{1}{n}\sum_{i=1}^n \ln(X_i) \text{ and } \ln(\bar{X}_H) = -\ln\left(\frac{1}{n}\sum_{i=1}^n \frac{1}{X_i}\right).$$

By Jensen's inequality,

$$\ln\left(\frac{1}{n}\sum_{i=1}^n X_i\right) \geq \frac{1}{n}\sum_{i=1}^n \ln(X_i),$$

which can be exponentiated to give the arithmetic mean-geometric mean inequality:

$$\underbrace{\frac{1}{n}\sum_{i=1}^n X_i}_{\bar{X}} \geq \underbrace{\left(\prod_{i=1}^n X_i\right)^{\frac{1}{n}}}_{\bar{X}_G}, \quad \text{thus } \bar{X} \geq \bar{X}_G.$$

Now comparing the harmonic with the geometric mean (and by Jensen's inequality):

$$-\ln\left(\frac{1}{n}\sum_{i=1}^n \frac{1}{X_i}\right) \leq -\frac{1}{n}\sum_{i=1}^n \ln\left(\frac{1}{X_i}\right) = \frac{1}{n}\sum_{i=1}^n \ln(X_i),$$

and by exponentiating both sides:

$$\underbrace{\frac{n}{\sum_{i=1}^n \frac{1}{X_i}}}_{\bar{X}_H} \leq \underbrace{\left(\prod_{i=1}^n X_i\right)^{\frac{1}{n}}}_{\bar{X}_G}, \quad \text{thus } \bar{X}_H \leq \bar{X}_G.$$

C Different Meanings of the Center

Median

Consider a first data set: 4, 6, 10, 100 ($n = 4$, even). Thus, the median rank is $r_M = \frac{1+4}{2} = 2.5$, the median is $\bar{X}_M = \frac{6+10}{2} = 8$ and $\sum_{i=1}^4 (r_i - r_M) = (1 - 2.5) + (2 - 2.5) + (3 - 2.5) + (4 - 2.5) = 0$.

For a second data set 4, 6, 10, 20, 100 ($n = 5$, odd), the median rank is $r_M = \frac{1+5}{2} = 3$, the median is $\bar{X}_M = 10$ and $\sum_{i=1}^{5}(r_i - r_M) = (1-3) + (2-3) + (3-3) + (4-3) + (5-3) = 0$. Thus, the median is the center of the distribution in terms of the counting observations: one half of the observations is on the left and one half is on the right of the median, no matter the value of the observations.

Arithmetic average

Consider again the second data set: $\bar{X}_A = \frac{4+6+10+20+100}{5} = 28$ and

$\sum_{i=1}^{5}(x_i - \bar{X}_A) = (4-28) + (6-28) + (10-28) + (20-28) + (100-28) = 0$.

Thus, the arithmetic average is the center of the distribution in terms of the deviations in absolute terms: the arithmetic mean is such that the absolute deviations on its right is compensate by the absolute deviations on its left. So, the center is defined in terms of the absolute deviations (or distances) between each value and the arithmetic average:

$$\underbrace{\underbrace{(4-28)}_{-24} + \underbrace{(6-28)}_{-22} + \underbrace{(10-28)}_{-18}\,\underbrace{(20-28)}_{-8}}_{-72} + \underbrace{(100-28)}_{+72}.$$

Geometric average

For the second data set: $\bar{X}_G = \sqrt[5]{4 \times 6 \times 10 \times 20 \times 100} = 13.69$ and

x_i	$\ln(x_i)$	$\ln(x_i) - \ln(\bar{X}_G)$
4	1.386	− 123.00%
6	1.792	− 82.45%
10	2.303	− 31.37%
20	2.996	37.94%
100	4.605	198.89%
	sum	0

Compounding percentage deviation means that:

$4 = 13.69 \times \exp(-123\%), \ldots, 100 = 13.69 \times \exp(198.89\%)$.

Thus, the geometric average is the value that balances the negative percentage deviations with the positive ones.

Harmonic average

x_i	$x_i - \bar{X}_H$	$w_i = 1/x_i$	$\frac{w_i}{\sum_{i=1}^{n} w_i}$	$(x_i - \bar{X}_H)\left(\frac{w_i}{\sum_{i=1}^{n} w_i}\right)$
4	− 4.671	0.250	0.434	− 2.025
6	− 2.671	0.167	0.289	− 0.772
10	1.329	0.100	0.173	0.231
20	11.329	0.050	0.087	0.982
100	91.329	0.010	0.017	1.584
		0.577	1	0

Thus, the harmonic average defines the center of the distribution in order that the weighted deviations on its left compensate the weighted deviations on its right. The weights are inversely proportional to the original values.

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Declarations

Conflict of interest The author declares that he has no conflict of interest.

Ethical approval No particular ethical approval was required for this study because it does not entail human participation or personal data.

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